International Journal of Theoretical Physics, Vol. 31, No. 10, 1992

Physical Content of Preparation-Question Structures and Brouwer–Zadeh Lattices

Gianpiero Cattaneo[1] **and Giuseppe Nisticò**[2]

Received February 11, 1992

We give a criterion to compare the physical content of different mathematical structures derived from a preparation-question structure. Then this criterion is used in order to compare the physical content of the (Jauch-Piron's) property lattice with the physical content of the poset of testable properties. We prove that for complete preparation-question structures these two structures carry the same physical content; moreover the set of testable properties has the algebraic structure of the Brouwer-Zadeh lattice. For more general preparation-question structures the physical content of the poset of testable property can be larger than that of the property lattice. Physically relevant examples of the possible cases are given.

1. INTRODUCTION

The Jauch–Piron approach to quantum mechanics (Jauch, 1983; Piron, 1976) uses primitive concepts such as *question* and *truth of questions* to get the property lattice $\mathscr{L}$ as a derived mathematical structure. Some axioms, explicitly or implicitly stated in the Jauch–Piron theory, appear as statements of obvious validity, on the basis of the physical interpretation of primitive concepts. We call these axioms *basic specific axioms* of the theory.

A *preparation-question structure*, denoted by $\langle \mathscr{S}, \mathscr{Q}; I, \pi, {}^{\sim}; T \rangle$, according to Cattaneo *et al.* (1988), is a mathematical structure whose terms formalize all primitive concepts, and only these, of the Jauch–Piron theory, and which is equipped with the basic specific axioms. Other axioms may be added so as to obtain more specific theories. We call these latter *specific peculiar axioms*. For instance, quantum mechanics with superselection rules may be recovered from a preparation-question structure requiring Piron's axioms

[1]Dipartimento di Scienze dell'Informazione, Università di Milano, Milan, Italy.
[2]Dipartimento di Matematica, Università della Calabria, Rende, Italy.

C, P, A or Aerts' (1981*a*, 1982) axioms 1–5; indeed, if one of these sets of axioms holds, then the property lattice $\mathscr{L}$ turns out to be isomorphic to the direct union of a suitable family of lattices $(\mathscr{M}(\mathscr{H}_i))$ [here $\mathscr{M}(\mathscr{H})$ denotes the lattice of closed subspaces of a generalized Hilbert space $\mathscr{H}$]. Thus, the property lattice $\mathscr{L}$ is a privileged mathematical structure derived from a preparation-question structure. However, several mathematical structures, different from the property lattice, can be derived from a preparation-question structure (without specific peculiar axioms). In particular, in Section 4 we define the poset L of *testable properties* and a structure $L(\Sigma, \#)$ which turns out to be a *BZ-lattice*, a category of mathematical structures introduced in Cattaneo and Nisticò (1989*a*).

What is the physical relevance of these new structures? Are they more or less relevant than the property lattice $\mathscr{L}$? In this paper we will try to answer these questions and therefore deal with the problem of *comparing or measuring* the physical content of mathematical structures derived from a preparation-question structure. In so doing we assume that the *whole physical content of a theory based on preparation-question structures is carried by the preparation-question structure* $\langle \mathscr{S}, \mathscr{Q}; I, \pi, {}^{\sim}; T \rangle$ *itself*, because all possible physical facts are collected in it. Our point of view is that a purely mathematical manipulation cannot create physical information, so that every structure derived from a preparation-question structure cannot carry a larger physical content. Thus, in general, in the passage from the preparation-question structure to a derived structure we expect a certain loss of physical information to occur. However, to give a concrete form to our ideas, in Section 4 we give a precise criterion (Definition 4.1) to compare the physical content of two mathematical structures. By using this criterion and the mathematical properties of a BZ-lattice we can prove that the physical content of the property lattice $\mathscr{L}$ coincides with that of the poset L of testable properties if a specific peculiar axiom, named the *completeness axiom*, holds. In this case L turns out to be a BZ-lattice. Generally the physical content carried by L is larger than the physical content carried by the property lattice $\mathscr{L}$.

In Section 2 we introduce the notion of preparation-question structure (pqs). In Section 3 we outline Jauch and Piron's and Aerts' theories; moreover, we provide a mathematical model of a preparation-question structure which is based on a Hilbert space, and embodies quantum mechanics. In Section 4 we derive the poset L of testable properties and the BZ-lattice $L(\Sigma, \#)$ from a preparation-question structure. We show that, without peculiar specific axioms, the physical content carried by L is larger than that carried by $\mathscr{L}$. In Section 4.2 we introduce the completeness axiom. We show that if this axiom holds, L turns out to be a BZ-lattice isomorphic to $L(\Sigma, \#)$, and this enables us to state that $\mathscr{L}$ and L carry the same physical content.

However, we show in Section 4.3 that the completeness axiom does not hold in a pqs which describes two separated entities according to Aerts' (1981*a*, 1982) theory. We conclude that we cannot attribute the whole physical content of the theory to the property lattice $\mathscr{L}$ in general theories based on preparation-question structures.

2. GENERAL THEORY

The Jauch–Piron (JP) approach to quantum physics (Jauch, 1983; Piron, 1976) starts by introducing the primitive concepts of the theory, i.e., those terms having a direct physical interpretation. In the following items these concepts are presented and their physical interpretation is given.[3]

1. *Preparation procedures.* A *preparation procedure* of an entity S is any reproducible way of obtaining samples of the entity. We denote the set of preparation procedures of the entity S by $\mathscr{S}$ and single preparation procedures by letters x, y, z, etc. Since a preparation procedure $x\in\mathscr{S}$ is reproducible, it can also be used to obtain ensembles of arbitrary size of single samples prepared according to x.

2. *Questions.* By a *question* of an entity S we mean any experimental observation performable on any single sample of the entity and such that, each time it is carried out, the outcome is always interpretable either as "yes" or as "no." We denote questions by Greek letters α, β, γ, etc., and by $\mathscr{Q}$ the set of the questions of the entity.

3. *Truth and falsehood of a question.* A question $\alpha\in\mathscr{Q}$ is said to be *true* (resp. *false*) in $x\in\mathscr{S}$ if any performance of the question α with an arbitrary sample of the entity prepared according to x yields the outcome "yes" (resp., "no") with certainty. This concept is properly formalized by means of a relation $T\subseteq\mathscr{S}\times\mathscr{Q}$ (resp., $F\subseteq\mathscr{S}\times\mathscr{Q}$). Thus, if the question α is true (resp., false) in $x\in\mathscr{S}$, we write $T(x, \alpha)$ [resp., $F(x, \alpha)$].

Sometimes the following concept also appears in the JP approach.

(M1) *Occurrence of a question.* We say that a question α occurs (resp., does not occur) if the performance of α yields the outcome "yes" (resp., "no").

However, it must be noticed that the concept (M1) cannot be formalized in the now proposed JP framework; it must be regarded as a *metatheoretical* concept rather than a primitive one of the theory.

Definition 2.1. Let α be a question of an entity S; subsets

$$\mathscr{S}_T(\alpha)=\{x\in\mathscr{S}|T(x, \alpha)\} \qquad \text{and} \qquad \mathscr{S}_F(\alpha)=\{x\in\mathscr{S}|F(x, \alpha)\}$$

[3]In our presentation we explicitly formalize the concept of preparation procedure, which the JP approach uses without formalizing it.

will be called the certainly true and the certainly false domain of the question α, respectively.

The following statements are assumed as *basic specific axioms* of the theory.

A1 For every $\alpha \in \mathscr{Q}$, $\mathscr{S}_T(\alpha) \cap \mathscr{S}_F(\alpha) = \varnothing$.

A2 A question $I \in \mathscr{Q}$ exists such that $\mathscr{S}_T(I) = \mathscr{S}$. This trivial question is called the *certain* question.

A3 Given an arbitrary family of questions $\{\alpha_i\} \subseteq \mathscr{Q}$, a question $\pi_i \alpha_i \in \mathscr{Q}$ exists and is called the *product* of all α_i such that

$$\mathscr{S}_T(\pi_i \alpha_i) = \bigcap_i \mathscr{S}_T(\alpha_i) \qquad \text{and} \qquad \mathscr{S}_F(\pi_i \alpha_i) = \bigcap_i \mathscr{S}_F(\alpha_i)$$

The product of two questions α and β will be also denoted by $\alpha \cdot \beta$ or $\alpha\beta$.

A4 A mapping

$$^\sim\colon\ \mathscr{Q} \mapsto \mathscr{Q}, \quad \alpha \mapsto \alpha^\sim$$

exists such that (i) for every $\alpha \in \mathscr{Q}$, $\mathscr{S}_T(\alpha^\sim) = \mathscr{S}_F(\alpha)$, $\mathscr{S}_F(\alpha^\sim) = \mathscr{S}_T(\alpha)$, and $\alpha^{\sim\sim} = \alpha$; and (ii) for every family $\{\alpha_i\} \subseteq \mathscr{Q}$, we have

$$(\pi_i \alpha_i)^\sim = \pi_i \alpha_i^\sim$$

The question $\alpha^\sim$ is call the *inverse* of α.

Definition 2.2. We denote the question $I^\sim$ by O, and call it the *absurd* question.

Question I is nothing else but a question testing the *presence* of the entity S; therefore, it is always true, so its inverse O is always false. The product question $\pi_i \alpha_i$ is performed by carrying out one of the questions α_i, which is chosen in an arbitrary way, at random or not, and by attributing the outcome so obtained to $\pi_i \alpha_i$. Given a question α, its inverse $\alpha^\sim$ is the question performed by carrying out α and attributing to $\alpha^\sim$ the outcome "no" if "yes" is obtained for α, and "yes" if "no" is obtained for α.

We call *preparation-question structure* (pqs) any pair $(\mathscr{S}, \mathscr{Q})$ endowed with a relation T, a unary operation $^\sim$, and an infinitary operation π on $\mathscr{Q}$ such that statements A1–A4 hold; we denote a pqs by

$$\langle \mathscr{S}, \mathscr{Q}; I, \pi, {}^\sim; T \rangle$$

3. THE JAUCH–PIRON AND AERTS THEORIES

In the JP approach to quantum physics, a pqs $\langle \mathscr{S}, \mathscr{Q}; I, \pi, {}^\sim; T \rangle$ may be associated to every entity S. Then, JP theory proceeds by defining the main derived structure, the *property lattice* $\mathscr{L}$. A set of specific peculiar axioms, namely axioms C, P, and A in Piron (1976), are added. By using these axioms Piron proves that an irreducible property lattice $\mathscr{L}$, whose dimension is at least 4, is isomorphic to the lattice of closed subspaces of a generalized Hilbert space $\mathscr{H}$. Thus, JP theory provides a conceptual and axiomatic foundation of customary quantum mechanics.

However, physical theories other than JP's can be formulated by choosing specific peculiar axioms which are different from C, P, A. As one example, we quote Aerts' theory. These different theories have a common feature: the physical interpretation of the whole formalism is possible only by using rules given in items 1–3 of Section 2; i.e., such an interpretation is based on the physical concepts carried by the underlying pqs.

In Section 3.1 new notions, derived from a pqs, are introduced and the formalism is developed without introducing further axioms besides A1–A4 which characterize every pqs. In Sections 3.2 and 3.3 we outline the JP and Aerts theories, respectively. In Section 3.4 we exhibit the Hilbertian model of pqs.

3.1. The pqs Formalism

Given a pqs $\langle \mathscr{S}, \mathscr{Q}; I, \pi, {}^{\sim}; T\rangle$, following Piron (1976), we define a quasiorder relation on $\mathscr{Q}$ by

$$\alpha < \beta \qquad \text{iff} \quad \mathscr{S}_T(\alpha) \subseteq \mathscr{S}_T(\beta) \tag{QO1}$$

This quasiorder relation induces the following equivalence relation $\approx$ on $\mathscr{Q}$:

$$\begin{aligned} \alpha \approx \beta \qquad &\text{iff} \quad \alpha < \beta \text{ and } \beta < \alpha \\ &\text{iff} \quad \mathscr{S}_T(\alpha) = \mathscr{S}_T(\beta) \end{aligned} \tag{EQ1}$$

An equivalence class $[\alpha]_\approx$ generated by means of the equivalence relation (EQ1) is called a *property*. The set of all properties of an entity S, i.e., the quotient set $\mathscr{Q}/_\approx$, will be denoted by $\mathscr{L}$. The following theorem, proved by Piron (1976), allows us to say that $\mathscr{L}$ is the *property lattice* of the entity S.

Theorem 3.1. The quotient set $\mathscr{L} = \mathscr{Q}/_\approx$ endowed with the order relation

$$a \le b \qquad \text{iff} \quad \alpha < \beta, \quad \forall \alpha \in a \quad \text{and} \quad \forall \beta \in b \tag{O2}$$

is a complete lattice where, denoting the l.u.b. and the g.l.b. by $\vee$ and $\wedge$, respectively:

(i) $\bigwedge_i a_i = [\pi_i \alpha_i]_\approx$, with $\alpha_i \in a_i$
(ii) $\bigvee_{a \in \mathscr{L}} a \equiv \mathbb{1} = [I]_\approx$, $\bigwedge_{a \in \mathscr{L}} a \equiv \mathbb{0} = [O]_\approx$

A property $a \in \mathcal{L}$ of an entity is said to be *actual* in the preparation procedure $x \in \mathcal{S}$ iff $x \in \mathcal{S}_T(\alpha)$ for one, and therefore all, $\alpha \in a$.

According to Jauch and Piron (1969) and Aerts (1982), we identify a *state* of an entity with the set of its actual properties. If the entity is prepared according to a preparation procedure $x \in \mathcal{S}$, the set of all its actual properties is

$$\sigma(x) = \{a \in \mathcal{L} | x \in \mathcal{S}_T(\alpha),\ \alpha \in a\}$$

Then we denote by $p(x)$ the property defined as

$$p(x) = \bigwedge_{a \in \sigma(x)} a \in \mathcal{L}$$

Note that for every $x \in \mathcal{S}$, $p(x) \in \sigma(x)$ and $p(x) \neq \mathbb{0}$. Since $a \in \sigma(x)$ iff $p(x) \leq a$, we have that

$$\sigma(x) = \{a \in \mathcal{L} | p(x) \leq a\}$$

and $p(x)$ is called the *state* of the entity prepared according to x. The set of all states of an entity S will be denoted by Σ.

Given $\alpha \in \mathcal{Q}$ and $a \in \mathcal{L}$, we define the following subsets of Σ:

$$\begin{aligned}
\Sigma_T(\alpha) &= \{p \in \Sigma | \exists x \in \mathcal{S}_T(\alpha), p(x) = p\} \\
&= \{p(x) \in \Sigma | x \in \mathcal{S}_T(\alpha)\} \\
\Sigma_F(\alpha) &= \{p \in \Sigma | \exists x \in \mathcal{S}_F(\alpha), p(x) = p\} \\
&= \{p(x) \in \Sigma | x \in \Sigma_F(\alpha)\} \\
\Sigma_T(a) &= \{p \in \Sigma | p \in \Sigma_T(\alpha),\ \alpha \in a\}
\end{aligned}$$

It is straightforward to prove the following proposition.

Proposition 3.1. In a pqs $\langle \mathcal{S}, \mathcal{Q}; I, \pi, {}^{\sim}; T \rangle$ the following statements hold.

(i) Let $\alpha \in \mathcal{Q}$; then $x \in \mathcal{S}_T(\alpha)$ iff $p(x) \in \Sigma_T(\alpha)$.
(ii) Let $\alpha, \beta \in \mathcal{Q}$; then $\alpha < \beta$ iff $\Sigma_T(\alpha) \subseteq \Sigma_T(\beta)$.
(iii) Let $\alpha, \beta \in \mathcal{Q}$; then $\alpha \approx \beta$ iff $\Sigma_T(\alpha) = \Sigma_T(\beta)$; therefore, $\Sigma_T(a) = \Sigma_T(\alpha)$, $\alpha \in a$.
(iv) Let $a, b \in \mathcal{L}$; then $a \leq b$ iff $\Sigma_T(a) \subseteq \Sigma_T(b)$.
(v) Let $a, b \in \mathcal{L}$; $a = b$ iff $\Sigma_T(a) = \Sigma_T(b)$.

We define the subset $\mathcal{L}(\Sigma, \mathcal{Q})$ of $\mathcal{P}(\Sigma)$ as follows:

$$\begin{aligned}
\mathcal{L}(\Sigma, \mathcal{Q}) &= \{\Sigma_T(a) | a \in \mathcal{L}\} \\
&= \{\Sigma_T(\alpha) | \alpha \in \mathcal{Q}\}
\end{aligned}$$

By using Proposition 3.1, it is straightforward to prove that $\mathscr{L}(\Sigma, \mathscr{Q})$, equipped with the set-theoretic inclusion partial ordering is a lattice isomorphic to the property lattice $\mathscr{L}$ via the mapping

$$\mathrm{ext}_1: \quad \mathscr{L} \mapsto \mathscr{L}(\Sigma, \mathscr{Q}), \qquad a \mapsto \mathrm{ext}_1(a) = \Sigma_T(a)$$

An irreflexive and symmetric binary relation, called a *preclusivity relation*, can be defined on Σ as follows[4]

$$p\#q \quad \text{iff} \quad \exists \alpha \in \mathscr{Q} \quad \text{such that} \quad p \in \Sigma_T(\alpha) \quad \text{and} \quad q \in \Sigma_F(\alpha) \qquad \text{(P)}$$

We now collect some results which are related to any irreflexive and symmetric binary relation $\#$ on a nonempty set Σ. Let A, B be two subsets of Σ; we write $A(\#)B$ if $a\#b$, for every $a \in A$ and $b \in B$. Given any subset $A \subseteq \Sigma$, we set

$$A^{\#} = \{p \in \Sigma \mid p\#q, \forall q \in A\}$$

and so if $A(\#)B$, then $B \subseteq A^{\#\#}$.

Proposition 3.2. Let A, B be subsets of Σ. Then, the following statements hold.

(i) $A \subseteq A^{\#\#}$.
(ii) $A \subseteq B$ implies $B^{\#} \subseteq A^{\#}$.
(iii) $A \cap A^{\#} = \varnothing$.

Furthermore, for every subset A of Σ we have:

(iv) $A^{\#} = A^{\#\#\#}$.

Moreover, for every family $\{A_i\}$ of subsets of Σ we have:

(v) $(\bigcup_i A_i)^{\#} = \bigcap_i A_i^{\#}$.

Proof. Straightforward.

Definition 3.1. A subset A of Σ will be said to be $\#$-closed if $A = A^{\#\#}$. We denote by $\mathscr{L}(\Sigma, \#)$ the set of all $\#$-closed subsets of Σ.

The set $\mathscr{L}(\Sigma, \#)$ is never empty since the trivial subsets $\varnothing$ and Σ are $\#$-closed. The proofs of the following propositions are straightforward.

Proposition 3.3. The structure $(\mathscr{L}(\Sigma, \#), \subseteq, ^{\#})$ is an orthocomplemented, complete lattice with respect to set-theoretic inclusion $\subseteq$ and to the orthocomplementation mapping

$$^{\#}: \quad \mathscr{L}(\Sigma, \#) \mapsto \mathscr{L}(\Sigma, \#), \qquad A \mapsto A^{\#}$$

[4]This relation was called "orthogonality" by Aerts (1982); we prefer the name "preclusivity" to avoid confusion with a possible orthogonality definable on $\mathscr{L}$.

Moreover, for every family $\{A_i\}$ of $\#$-closed subsets the greatest lower bound (g.l.b.) and the lowest upper bound (l.u.b.) are given by:

(i) $\text{g.l.b.}\{A_i\} = \bigcap_i A_i$. (ii) $\text{l.u.b.}\{A_i\} \equiv \bigvee_i A_i = (\bigcup_i A_i)^{\#\#}$.

Furthermore, the generalized de Morgan's laws hold:

(iii) $(\bigvee_i A_i)^{\#} = \bigcap_i A_i^{\#}$.
(iv) $(\bigcap_i A_i)^{\#} = \bigvee_i A_i^{\#}$.

Finally, for any subset A of Σ the subset $A^{\#\#}$ is $\#$-closed, i.e., it is an element of $\mathscr{L}(\Sigma, \#)$, containing A, and such that

$$A^{\#\#} = \bigcap \{B \in \mathscr{L}(\Sigma, \#) : A \subseteq B\}$$

thus, $A^{\#\#}$ is the $\#$-closure of A.

Proposition 3.4. For every $\alpha \in \mathscr{Q}$:

(i) $\Sigma_T(\alpha) \subseteq \Sigma_F(\alpha)^{\#}$ and $\Sigma_F(\alpha) \subseteq \Sigma_T(\alpha)^{\#}$.
(ii) $\Sigma_T(\alpha)^{\#\#}(\#)\Sigma_F(\alpha)^{\#\#}$.

The notion of primitive question, expressed by the following definition, is due to Aerts (1982).

Definition 3.2. Let $\langle \mathscr{S}, \mathscr{Q}; I, \pi, {}^{\sim}; T \rangle$ be a pqs. A question $\alpha \in \mathscr{Q}$ is said to be primitive if $\Sigma_T(\alpha) = \Sigma_F(\alpha)^{\#}$ and $\Sigma_F(\alpha) = \Sigma_T(\alpha)^{\#}$. We shall denote the set of all primitive questions by $\mathscr{P}$.

Remark 1. Our definition is nothing else but the translation, in terms of preclusivity relation, of the concept of primitive question is defined by Aerts (1982): "A question α is said to be *primitive* if (α is true in every state q orthogonal to all states in which $\alpha^{\sim}$ is true) and ($\alpha^{\sim}$ is true in every state p orthogonal to all states in which α is true)."

Proposition 3.5. Let $\langle \mathscr{S}, \mathscr{Q}; I, \pi, {}^{\sim}; T \rangle$ be a pqs. The following statements hold.

(i) A question α is primitive iff $\alpha^{\sim}$ is primitive.
(ii) If α and β are primitive questions, then $\alpha \approx \beta$ iff $\alpha^{\sim} \approx \beta^{\sim}$.
(iii) A question of the form

$$\pi_i \alpha_i$$

is primitive iff $\alpha_{i_1} \approx \alpha_{i_2}$ and α_i is primitive for all i_1, i_2, and i.

(iv) If α is a primitive question, then $\Sigma_T(\alpha)$ and $\Sigma_F(\alpha)$ are $\#$-closed subsets of Σ.

3.2. Jauch–Piron Theory

Before introducing the specific peculiar axioms of JP theory, we recall some definitions.

Compatible complement. A property $b \in \mathscr{L}$ is a compatible complement of a property $a \in \mathscr{L}$ if $a \vee b = \mathbb{1}$, $a \wedge b = \mathbb{0}$, and there exists $\alpha \in a$ such that $\alpha^{\sim} \in b$. We denote a compatible complement of a by a'.

Covering relation. Given two elements $a, b \in \mathscr{L}$, we say that a covers b, and we write $a \mathscr{C} b$ if for every $c \in \mathscr{L}$, $b \leq c \leq a$ implies $c = b$ or $c = a$.

Atom. An element $p \in \mathscr{L}$ is an atom of the lattice $\mathscr{L}$ if $p \neq \mathbb{0}$ and $p \mathscr{C} \mathbb{0}$.

The following axioms C, P, and A constitute the set of specific peculiar axioms of Piron's theory.

Axiom C. Every property $a \in \mathscr{L}$ has at least one compatible complement.

Axiom P. If $a, b \in \mathscr{L}$ and $a \leq b$, then the sublattice of $\mathscr{L}$ generated by $\{a, b, a', b'\}$ is a Boolean lattice.

Axiom A. (A_1) For every property $a \in \mathscr{L}$ there exists an atom p such that $p \leq a$; (A_2) if p is an atom and b a property such that $p \wedge b = \mathbb{0}$, then $(p \vee b) \mathscr{C} b$ (covering law).

It follows from axioms C and P that the compatible complement a' of every property $a \in \mathscr{L}$ is unique, so that $\langle \mathscr{L}, \leq, ' \rangle$ turns out to be an orthocomplemented and weakly modular, i.e., orthomodular, atomic complete lattice in which the covering law holds. The following theorem is a very important result in JP theory.

Theorem 3.2 (Piron). Every irreducible, orthomodular, atomic, and complete lattice of dimension at least 4 in which the covering law holds is isomorphic to the lattice $\mathscr{L}(\mathscr{H})$ of all closed subspaces of a generalized Hilbert space, or, equivalently, to the lattice $\mathscr{E}(\mathscr{H})$ of its orthogonal projections.

3.3. Aerts' Theory

Aerts' (1981*a*, 1982) approach to quantum physics differs from the JP theory only in a different choice of the specific peculiar axioms. Aerts introduces five axioms.

Axiom 1. The set $\mathscr{Q}$ of questions of an entity S is generated by the set $\mathscr{P}$ of primitive questions of S by means of the unary operation $^{\sim}$ and the initary operation π.

Axiom 2. For every state $p \in \Sigma$, there exists a question $\alpha_p \in \mathscr{Q}$ such that $\Sigma_T(\alpha_p) = \{p\}^{\#}$.

Axiom 3. Every state $p \in \Sigma$ is an atom of the property lattice $\mathscr{L}$.

Axioms 4 and 5 of Aerts' theory coincide with axiom P and axiom A_2 of the JP theory, respectively.

Let us compare the two sets of axioms. We see that axiom 3 is equivalent to axiom A_1; thus, we get Aerts' theory from JP's by replacing axiom C with axioms 1 and 2.

By using axioms 1–5, Aerts proves that the property lattice $\mathscr{L}$ is, as in JP theory, an orthomodular, atomic complete lattice in which the covering law holds; therefore, Piron's Theorem 3.2 applies to Aerts' property lattice, too.

3.4. Hilbert Space Model of pqs

Let $\mathscr{H}$ be a Hilbert space. We denote by $\mathscr{F}(\mathscr{H})$ the set of all linear bounded self-adjoint operators α on $\mathscr{H}$ such that $0 \leq \alpha \leq 1$, and by $\mathscr{E}(\mathscr{H}) \subseteq \mathscr{F}(\mathscr{H})$ the set of all (orthogonal) projection operators on $\mathscr{H}$. Furthermore, we denote the composition of mappings by $\circ$, and for every $\alpha \in \mathscr{F}(\mathscr{H})$ we call E_α the resolution of the identity that belongs to α; finally, for every trace class operator ρ we denote the trace of ρ by $\mathrm{Tr}[\rho]$.

Then, we now give the following definitions.

1. We denote the subset of $\mathscr{E}(\mathscr{H})$ of all the projections over one-dimensional subspaces of $\mathscr{H}$ by $\mathscr{S}(\mathscr{H})$; an element $x \in \mathscr{S}(\mathscr{H})$ is called a Hilbertian preparation procedure.
2. $\mathscr{F}(\mathscr{H})$ is called the set of Hilbertian questions.
3. I denotes the identity operator on $\mathscr{H}$.
4. The sign $\tilde{}$ denotes the mapping

$$\tilde{}: \quad \alpha \in \mathscr{F}(\mathscr{H}) \mapsto \alpha^{\sim} = (I - \alpha) \in \mathscr{F}(\mathscr{H})$$

5. The sign Π denotes the mapping

$$\Pi: \quad \mathscr{P}(\mathscr{F}(\mathscr{H})) \mapsto \mathscr{F}(\mathscr{H}),$$

$$a \mapsto \prod_{\alpha \in a} \alpha = \frac{1}{2}\left(\bigcup_{\alpha \in a} E_\alpha(\{1\}) + \left(\bigcap_{\alpha \in a} E_\alpha(\{0\})\right)^{\sim}\right) \in \mathscr{F}(\mathscr{H})$$

(a being any family of operators of $\mathscr{F}(\mathscr{H})$) with $\mathscr{P}(\mathscr{F}(\mathscr{H}))$ the power set of $\mathscr{F}(\mathscr{H})$ and $\bigcap$ the usual meet in the lattice of all the (orthogonal) projection operators on $\mathscr{H}$; we explicitly note that generally $\prod_{\alpha \in a} \alpha$ is not a projection, even when $a \subseteq \mathscr{E}(\mathscr{H})$.

6. T denotes the predicate "the trace is 1"; more precisely, for every $x \in \mathscr{S}(\mathscr{H})$, $\alpha \in \mathscr{F}(\mathscr{H})$

$$T(x, \alpha) \qquad \text{iff} \quad \mathrm{Tr}[x \circ \alpha] = 1$$

(equivalently, $\mathrm{Tr}[x \circ E_\alpha(\{1\})] = 1$).

According to the above definitions, we state the following mathematical propositions (whose proofs are straightforward).

P.1. Let $x \in \mathscr{S}(\mathscr{H})$, $\alpha \in \mathscr{F}(\mathscr{H})$; then, the properties $\mathrm{Tr}[x \circ \alpha] = 1$ and $\mathrm{Tr}[x \circ \alpha^\sim] = 1$ are mutually exclusive.

P.2. Let $x \in \mathscr{S}(\mathscr{H})$; then, $\mathrm{Tr}[x \circ I] = 1$.

P.3. Let $x \in \mathscr{S}(\mathscr{H})$, $a \in \mathscr{P}(\mathscr{F}(\mathscr{H}))$; then

$$\mathrm{Tr}[x \circ \alpha] = 1 \text{ for every } \alpha \in a$$

$$\text{iff } \mathrm{Tr}\left[x \circ \left(\frac{1}{2} \bigcap_{\alpha \in a} E_\alpha(\{1\}) + \frac{1}{2}\left(\bigcap_{\alpha \in a} E_\alpha(\{0\})\right)^\sim\right)\right] = 1$$

$$\mathrm{Tr}[x \circ \alpha] = 0 \text{ for every } \alpha \in a$$

$$\text{iff } \mathrm{Tr}\left[x \circ \left(\frac{1}{2} \bigcap_{\alpha \in a} E_\alpha(\{1\}) + \frac{1}{2}\left(\bigcap_{\alpha \in a} E_\alpha(\{0\})\right)^\sim\right)\right] = 0$$

P.4. For every $\alpha \in \mathscr{F}(\mathscr{H})$, $\alpha^{\sim\sim} = \alpha$, and for every $a \in \mathscr{P}(\mathscr{F}(\mathscr{H}))$, $(\prod_{\alpha \in a} \alpha)^\sim = (\prod_{\alpha \in a} \alpha^\sim)$.

Thus, in the structure $\langle \mathscr{S}(\mathscr{H}), \mathscr{F}(\mathscr{H}); \Pi, {}^\sim; T \rangle$ statements A1–A4 in Section 2 can be proved as theorems; i.e., this structure is a pqs.

Remark 2. In the Hilbertian model of pqs, the property lattice $\mathscr{L}(\mathscr{H})$ is isomorphic to the lattice $\mathscr{E}(\mathscr{H})$ of all orthogonal projections of $\mathscr{H}$. Now, by the spectral theorem, every $\alpha \in \mathscr{F}(\mathscr{H})$ can be expressed in terms of orthogonal projections, i.e.,

$$\alpha = \int \lambda \, dE_\lambda$$

where $E_\lambda \in \mathscr{E}(\mathscr{H})$. Thus, a feature of the Hilbertian model of pqs is that the set of Hilbertian questions may be reconstructed from the property lattice. This enables us to say that, in the Hilbertian pqs, the whole physical content of the theory is carried by the property lattice.

4. PHYSICAL CONTENT OF A THEORY BASED ON PQS

Quantum mechanics in Hilbert spaces (HQM), as formulated by von Neumann (1955), for instance, is regarded by most physicists as the "right" theory to describe quantum phenomena. One can say that HQM has been successful in fulfilling this task. The rather technical formalism of HQM, so far away from human experience, finds an acceptable justification in some foundational approaches to quantum mechanics, such as the JP and Aerts theories outlined in the previous section. Indeed, in both approaches, the property lattice $\mathscr{L}$ turns out to be isomorphic, because of Piron's Theorem 3.2, to the lattice of closed subspaces of a generalized Hilbert space, thus recovering the formalism of HQM on the conceptual ground formalized by a pqs. For this reason HQM can be called a *realization* of Piron's theory.

However, we stress that HQM is a *partial* realization of the whole formalism stemming from a pqs; indeed, HQM realizes only the *derived* structure $\mathscr{L}$. Nevertheless, we notice that, in most foundational works, physicists are concerned with the property lattice $\mathscr{L}$, while completely neglecting the underlying pqs. This happens also when one needs to formalize physical concepts of fundamental importance, such as *compatibility* and *composite entities*. Piron himself, for instance, introduces the compatibility as a binary relation on the property lattice. The problem of the description of composite entities is viewed as a "lattice-theoretic problem" by Aerts and Daubechies (1978*a*,*b*, 1979), Aerts (1981*b*), and Nash and Joshi (1987*a*,*b*) [there are, however, valuable exceptions: let us just quote Aerts' theory of separated entities formulated at the level of the preparation-question structure, rather than at the level of the property lattice (Aerts, 1981*a*, 1982)].

Now, this is a correct view for those who believe that the whole physical content of the theory is carried by the property lattice. Moreover, the fact that the "right" quantum physics, HQM, with all its limitations, is a realization of the property lattice seems to justify this view. Still, as stated in the final remark of the previous section, it is a matter of fact that, in the Hilbertian model of JP theory presented in Section 3.4, the entire pqs $\langle \mathscr{S}(\mathscr{H}), \mathscr{F}(\mathscr{H}); I, \Pi, {}^{\sim}; T\rangle$ may be reconstructed from the lattice $\mathscr{L}$. Hence, there are several good reasons pointing to the idea that the only theoretical structure of physical interest is the property lattice $\mathscr{L}$.

We think, however, that serious doubts can be raised against this view. The origin of these doubts lies within the results of some interesting theoretical researches about composite entities. Indeed, it seems that some physical composite entities cannot be described by means of a Hilbertian model of a pqs. In Aerts (1981*a*, 1982), for instance, it is claimed that customary quantum mechanics based on Hilbert spaces cannot describe an entity consisting of two "separated entities." In particular, according to Aerts, axioms 4 and 5 (or axioms P and A_2 of JP theory) cannot hold in a pqs describing an entity consisting of two separated entities individually described by HQM. Therefore, for such an entity Piron's Theorem 3.2 does not apply, and so the main reason for considering $\mathscr{L}$ the carrier of the whole physical content of the theory fails. Moreover, Pulmannová (1985) and Raggio (1991) proved results analogous to Aerts' in the frameworks of quantum logics and W^*-algebras, respectively.

In order to explore whether the property lattice $\mathscr{L}$ carries the whole physical content of a theory based on pqs, we must clarify the meaning of *physical content* carried by a mathematical structure derived from a pqs. We give the following definition.

Definition 4.1. Let $\mathscr{A}$ and $\mathscr{B}$ be two mathematical structures derived from the same pqs $\langle \mathscr{S}, \mathscr{Q}; I, \pi, {}^{\sim}; T\rangle$ describing an entity S. We say that $\mathscr{A}$ carries a larger physical content than $\mathscr{B}$ if there exists a canonical procedure to derive a mathematical structure $\mathscr{C}$ from $\mathscr{A}$ such that $\mathscr{B}$ is isomorphic to $\mathscr{C}$.

In other words, the physical content carried by $\mathscr{A}$ is greater than that carried by $\mathscr{B}$ if it is possible to recover $\mathscr{B}$ from $\mathscr{A}$.

4.1. Testable Properties

Given a pqs $\langle \mathscr{S}, \mathscr{Q}; I, \pi, {}^{\sim}; T\rangle$, we introduce a new quasiorder relation $\prec$ on $\mathscr{Q}$ as follows:

$$\alpha \prec \beta \quad \text{iff} \quad \Sigma_T(\alpha) \subseteq \Sigma_T(\beta) \quad \text{and} \quad \Sigma_F(\beta) \subseteq \Sigma_F(\alpha) \tag{QO2}$$

This relation is stronger than the quasiorder relation $<$ defined by (QO1) in Section 3; indeed

$$\alpha \prec \beta \quad \text{implies} \quad \alpha < \beta \tag{S1}$$

while, in general, the converse is not true. We may interpret the mathematical statement (S1) as saying that $\alpha \prec \beta$ provides more detailed physical information than $\alpha < \beta$. Now, the same procedure used in Section 3 to single out the structure $\mathscr{L}$ through the quasiorder relation $<$ may be used to single out a new structure L through the new quasiorder relation $\prec$. Hence, we define the following equivalence relation on $\mathscr{Q}$:

$$\begin{aligned} \alpha \equiv \beta \quad & \text{iff} \quad \alpha \prec \beta \text{ and } \beta \prec \alpha \\ & \text{iff} \quad \mathscr{S}_T(\alpha) = \mathscr{S}_T(\beta) \text{ and } \mathscr{S}_F(\alpha) = \mathscr{S}_F(\beta) \\ & \text{iff} \quad \Sigma_T(\alpha) = \Sigma_T(\beta) \text{ and } \Sigma_F(\alpha) = \Sigma_F(\beta) \end{aligned} \tag{EQ2}$$

Of course we have

$$\alpha \equiv \beta \quad \text{implies} \quad \alpha \approx \beta \tag{S2}$$

Definition 4.2. Let $\langle \mathscr{S}, \mathscr{Q}; I, \pi, {}^{\sim}; T\rangle$ be a pqs. A testable property is any equivalence class $e = [\alpha]_{\equiv} \in \mathscr{Q}/_{\equiv}$. The set of all testable properties is denoted by L.

It follows from (S2) that a single property $a=\{\alpha, \beta, \dots\}\in\mathscr{L}$ contains, in general, several testable properties $e=[\alpha]_{\equiv}$, $f=[\beta]_{\equiv}, \dots$. However, definition (EQ2) allows us to define, for every $e\in L$, two subsets of Σ as follows:

$$\Sigma_T(e)=\Sigma_T(\alpha) \quad \text{and} \quad \Sigma_F(e)=\Sigma_F(\alpha), \quad \alpha\in e$$

Note that L is a partially ordered set (poset) with respect to the relation

$$e\preceq f \quad \text{iff} \quad \alpha\prec\beta, \ \forall\alpha\in e \ \text{and} \ \beta\in f \tag{O2}$$

The poset L has the minimum element $\mathbf{0}=[O]_{\equiv}\subseteq\mathbb{0}$ and the maximum $\mathbf{1}=[I]_{\equiv}=\mathbb{1}$. In general, L is not a lattice with respect to $\preceq$. However, the inversion $\tilde{}$ and the product π of $\mathscr{Q}$ respectively induce the following mappings on L:

$$^{v}\colon \ L\mapsto L, \quad e=[\alpha]_{\equiv}\mapsto e^{v}=[\alpha^{\sim}]_{\equiv}$$

$$\Omega\colon \ \mathscr{P}(L)\mapsto L, \quad (e_j)_{j\in\mathscr{S}}\mapsto\Omega(e_j)_{j\in\mathscr{S}}=[\pi_j\alpha_j]_{\equiv}, \quad \text{where} \quad \alpha_j\in e_j$$

To simplify the notation, we write e_1e_2 instead of $\Omega(e_j, e_2)$.

We introduce two structures $L(\Sigma, \mathscr{Q})$ and $L(\Sigma, \#)$ which play, with respect to L, a role similar to that played by $\mathscr{L}(\Sigma, \mathscr{Q})$ and $\mathscr{L}(\Sigma, \#)$ with respect to $\mathscr{L}$.

$L(\Sigma, \mathscr{Q})$. We define the family $L(\Sigma, \mathscr{Q})\subseteq\mathscr{P}^2(\Sigma)$ as

$$L(\Sigma, \mathscr{Q})=\{(A, B), A, B\subseteq\Sigma|\exists e\in L \text{ such that } A=\Sigma_T(e) \text{ and } B=\Sigma_F(e)\}$$

A partial ordering relation $(A_1, B_1)\preceq(A_2, B_2)$ iff $A_1\subseteq A_2$ and $B_2\subseteq B_1$ is defined on $L(\Sigma, \mathscr{Q})$.

Proposition 4.1. The mapping

$$\text{ext}_2\colon \ L\mapsto L(\Sigma, \mathscr{Q}), \quad e\mapsto\text{ext}_2(e)=(\Sigma_T(e), \Sigma_F(e))$$

turns out to be an isomorphism between the posets L and $L(\Sigma, \mathscr{Q})$, i.e., ext_2 is a bijection such that:

(i) $\text{ext}_2(\mathbf{0})=(\varnothing, \Sigma)$ and $\text{ext}_2(\mathbf{1})=(\Sigma, \varnothing)$.
(ii) $\text{ext}_2(e)\preceq\text{ext}_2(f)$ iff $e\preceq f$.

$L(\Sigma, \#)$. Define the family $L(\Sigma, \#)$ by

$$L(\Sigma, \#)=\{(A, B)\in\mathscr{L}^2(\Sigma, \#)|A(\#)B\}$$

$L(\Sigma, \#)$ is endowed with the same partial order relation $\preceq$ defined on $L(\Sigma, \mathscr{Q})$. Now, the structure $(L(\Sigma, \#), \preceq)$ turns out to be a complete lattice, where for every family $\{(A_j, B_j)\}\in L(\Sigma, \#)$, we have:

(i) $\bigwedge_j (A_j, B_j)=(\bigwedge_j A_j, \bigvee_j B_j)$.
(ii) $\bigvee_j (A_j, B_j)=(\bigvee_j A_j, \bigwedge_j B_j)$.

(iii) $\bigwedge_{(A,B)\in L(\Sigma,\#)}(A, B)=(\varnothing, \Sigma)$ and $\bigvee_{(A,B)\in L(\Sigma,\#)}(A, B)=(\Sigma, \varnothing)$ [$\bigwedge$ and $\bigvee$ on the r.h.s. denote the g.l.b. and the l.u.b. of $\mathscr{L}(\Sigma, \#)$, respectively].

Furthermore, the two mappings

$$-: \; L(\Sigma, \#)\mapsto L(\Sigma, \#), \qquad -(A, B)=(B, A)$$

$$\sim: \; L(\Sigma, \#)\mapsto L(\Sigma, \#), \qquad \sim(A, B)=(B, A^{\#})$$

are defined on $L(\Sigma, \#)$. Such mappings satisfy the following properties:

(doc1) $-[-(A, B)]=(A, B)$
(doc2) $(A, B)\preceq(C, D)$ implies $-(C, D)\preceq-(A, B)$
(re) $(A, B)\preceq-(A, B)$ and $(C, D)\preceq-(C, D)$ imply $(A, B)\preceq(C, D)$
(woc1) $(A, B)\preceq\sim[\sim(A, B)]$
(woc2) $(A, B)\preceq(C, D)$ implies $\sim(C, D)\preceq\sim(A, B)$
(woc3) $(A, B)\wedge\sim(A, B)=(\varnothing, \Sigma)$
(in) $-[\sim(A,B)]=\sim[\sim(A, B)]$

Therefore, according to Cattaneo and Nisticò (1989*a*), the structure $\langle L(\Sigma, \#), (\varnothing, \Sigma), \preceq, -, \sim\rangle$ is a *BZ-lattice* with *half* element $(\varnothing, \varnothing)$.

So, we have derived the structure $(L, \preceq, \mathbf{0}, {}^{v}, \Omega)$ from a pqs $\langle\mathscr{S}, \mathscr{Q}; I, \pi, {}^{\sim}; T\rangle$. One may now ask the following question:

Is the physical content carried by L larger than the one carried by $\mathscr{L}$?

Physical intuition and statements (S1), (S2) suggest an affirmative answer. However, it is possible to recover the lattice $\mathscr{L}$ from the structure $(L, \preceq, \mathbf{0}, {}^{v}, \Omega)$, as the following proposition indicates (its proof is straightforward).

Proposition 4.2. Let $\langle\mathscr{S}, \mathscr{Q}; I, \pi, {}^{\sim}; T\rangle$ be a pqs. The following binary relation on the poset L

$$e<_L f \quad \text{iff} \quad e\preceq ef$$

is a quasiorder relation, which induces in L the equivalence relation

$$e\approx_L f \quad \text{iff} \quad e\preceq ef \;\text{ and }\; f\preceq ef$$

Therefore, the quotient set $\mathscr{L}(L):=L/_{\approx_L}$ is partially ordered by the relation $\leq_L$ defined by

$$l_1\leq_L l_2 \quad \text{iff} \quad e\in l_1 \;\text{ and }\; f\in l_2 \;\text{ imply }\; e<_L f$$

As a consequence, the following statements hold:

(i) For every $e, f\in L$, $\alpha, \beta\in\mathscr{Q}$, $e<_L f$ iff $\alpha\in e$ and $\beta\in f$ imply $\alpha<\beta$.

(ii) Let e, f, α, and β be as in (i); then $e\approx_L f$ iff $\alpha\in e$ and $\beta\in f$ imply $\alpha\approx\beta$.

(iii) For every $l \in \mathscr{L}(L)$, $\bigcup_{e\in l} e \in \mathscr{L}$.
(iv) For every $a \in \mathscr{L}$, $\lambda(a) := \{e \in L | e \subseteq a\} \in \mathscr{L}(L)$.
(v) For every $a, b \in \mathscr{L}$, $a \leq b$ iff $\lambda(a) \leq_L \lambda(b)$.
(vi) The poset $(\mathscr{L}(L), \leq_L)$ is a complete lattice.
(vii) The mapping $\lambda: \mathscr{L} \mapsto \mathscr{L}(L)$, $a \mapsto \lambda(a) = \{e \in l | e \subseteq a\}$, is an isomorphism between the complete lattices $(\mathscr{L}, \leq)$ and $(\mathscr{L}(L), \leq_L)$.

Therefore, given the structure $(L, \preceq, \mathbf{0}, {}^{\nu}, \mathbf{\Omega})$ induced from a pqs $\langle \mathscr{S}, \mathscr{Q}; I, \pi, {}^{\sim}; T\rangle$, it is always possible to single out a lattice $\mathscr{L}(L)$ *which is isomorphic to the property lattice* $\mathscr{L}$. We schematize this procedure as follows:

$$(L, \preceq, \mathbf{0}, {}^{\nu}, \mathbf{\Omega}) \rightarrow (\mathscr{L}(L), \leq_L) \leftrightarrow (\mathscr{L}, \leq) \qquad (L \rightarrow \mathscr{L})$$

Then, if we can get some information from the mathematical structure $(\mathscr{L}, \leq)$, we can get the same information from the structure $(L, \preceq, \mathbf{0}, {}^{\nu}, \mathbf{\Omega})$. In such a sense, according to Definition 4.1, we state that *the physical content carried by the structure L is larger than the one carried by the property lattice* $\mathscr{L}$.

When we describe an entity by taking into account only the property lattice $\mathscr{L}$, a certain amount of information on the entity contained in L can be lost. Such a loss of information does not occur if there exists a canonical procedure $(\mathscr{L} \rightarrow L)$ to deduce $(L, \preceq, \mathbf{0}, {}^{\nu}, \mathbf{\Omega})$ from the property lattice $(\mathscr{L}, \leq)$; in such a case we can affirm that L and $\mathscr{L}$ carry the same physical content. The following simple example shows that this procedure cannot be given for every pqs.

Example 4.1. Let $\mathscr{S}_1 = \{x_1, y_1, z_1\}$ be the set of preparation procedures for an entity S_1. Let $\mathscr{Q}_1$ be the set of its questions and $\mathscr{G}_1 = \{\alpha_1, \beta_1, \gamma_1\} \subseteq \mathscr{Q}_1$ be a set of (primitive) questions such that:

(i) $\mathscr{S}_T(\alpha_1) = \{x_1, y_1\}$, $\mathscr{S}_F(\alpha_1) = \varnothing$.
(ii) $\mathscr{S}_T(\beta_1) = \{x_1, z_1\}$, $\mathscr{S}_F(\beta_1) = \varnothing$.
(iii) $\mathscr{S}_T(\gamma_1) = \{y_1, z_1\}$, $\mathscr{S}_F(\gamma_1) = \varnothing$.

Furthermore, suppose that every $\alpha \in \mathscr{Q}_1$ is of the form

$$\alpha = \pi_j \alpha^{(j)}, \qquad \text{where} \quad \alpha^{(j)} \in \mathscr{G}_1 \cup \{O, I\} \quad \text{or} \quad (\alpha^{(j)})^{\sim} \in \mathscr{G}_1 \cup \{O, I\}$$

Then the set of states is $\Sigma_1 = \{p_1^x = p(x_1), p_1^y = p(y_1), p_1^z = p(z_1)\}$. The set $\mathscr{L}(\Sigma_1, \mathscr{Q}_1)$, isomorphic to the property lattice $\mathscr{L}_1 = \mathscr{Q}/_{\approx}$, is the Boolean lattice $\mathscr{L}(\Sigma_1, \mathscr{Q}_1) = \mathscr{P}(\Sigma_1)$, which consists of eight different elements. The set $L(\Sigma_1, \mathscr{Q}_1)$, isomorphic to $L_1 = \mathscr{Q}_1/_{\equiv}$ (Proposition 4.1) and since, for every $\alpha \in \mathscr{Q}_1$, $\Sigma_T(\alpha) = \varnothing$ or $\Sigma_F(\alpha) = \varnothing$, is given by

$$L(\Sigma_1, \mathscr{Q}_1) = \{(\varnothing, A) | A \in \mathscr{L}(\Sigma_1, \mathscr{Q}_1)\} \cup \{(A, \varnothing) | A \in \mathscr{L}(\Sigma_1, \mathscr{Q}_1)\}$$

It therefore consists of 15 different elements.

We now consider another entity S_2. Let $\mathscr{S}_2=\{x_2, y_2, z_2\}$ be the set of preparation procedures of S_2 and let $\mathscr{Q}_2$ be the set of its questions. Suppose that $\mathscr{G}_2=\{\alpha_2, \beta_2, \gamma_2, \delta_2\}\subseteq\mathscr{Q}_2$ is a set of primitive questions such that:

(i) $\mathscr{S}_T(\alpha_2)=\{x_2, y_2\}, \mathscr{S}_F(\alpha_2)=\varnothing$.
(ii) $\mathscr{S}_T(\beta_2)=\{x_2, z_2\}, \mathscr{S}_F(\beta_2)=\varnothing$.
(iii) $\mathscr{S}_T(\gamma_2)=\{y_2, z_2\}, \mathscr{S}_F(\gamma_2)=\varnothing$.
(iv) $\mathscr{S}_T(\delta_2)=\{x_2\}, \mathscr{S}_F(\delta_2)=\{y_2\}$.

As in the previous case, we assume that every $\alpha\in\mathscr{Q}_2$ is of the form

$$\alpha=\pi_j\alpha^{(j)}, \qquad \text{where } \alpha^{(j)}\in\mathscr{G}_2\cup\{O, I\} \quad \text{or} \quad (\alpha^{(j)})^{\sim}\in\mathscr{G}_2\cup\{O, I\}$$

The set Σ_2 of the states is $\Sigma_2=\{p_2^x=p(x_2), p_2^y=p(y_2), p_2^z=p(z_2)\}$. Since $\delta_2\approx\alpha_2\cdot\beta_2$, we have no further properties with respect to S_1. Then $\mathscr{L}(\Sigma_2, \mathscr{Q}_2)=\mathscr{P}(\Sigma_2)$ has eight elements, too. Let μ be the mapping

$$\mu\colon\ \Sigma_1\mapsto\Sigma_2, \qquad \mu(p_1^x)=p_2^x, \qquad \mu(p_1^y)=p_2^y, \qquad \mu(p_1^z)=p_2^z$$

The canonical extension of μ to a mapping $\mu:\mathscr{L}(\Sigma_1, \mathscr{Q}_1)\mapsto\mathscr{L}(\Sigma_2, \mathscr{Q}_2)$ turns out to be an isomorphism between the two complete lattices $\mathscr{L}(\Sigma_1, \mathscr{Q}_1)$ and $\mathscr{L}(\Sigma_2, \mathscr{Q}_2)$. We may therefore affirm that the property lattices $\mathscr{L}_1$ and $\mathscr{L}_2$ of the two respective entities S_1 and S_2 are isomorphic.

On the other hand, $\operatorname{ext}_2([\delta_2]_\equiv)=(\{p_2^x\}, \{p_2^y\})$, and the set $L(\Sigma_2, \mathscr{Q}_2)$, isomorphic to $L_2=\mathscr{Q}_2/_\equiv$, is given by

$$\begin{aligned} L(\Sigma_2, \mathscr{Q}_2)=&\{(\varnothing, A)|A\in\mathscr{L}(\Sigma_2, \mathscr{Q}_2)\}\\ &\cup\{(A, \varnothing)|A\in\mathscr{L}(\Sigma_2, \mathscr{Q}_2)\}\cup\{(\{p_2^x\}, \{p_2^y\}), (\{p_2^y\}, \{p_2^x\})\}\end{aligned}$$

One sees that it consists of 17 different elements, whereas $L(\Sigma_1, \mathscr{Q}_1)$ has only 15.

We thus conclude that, although S_1 and S_2 have isomorphic property lattices $\mathscr{L}_1$ and $\mathscr{L}_2$, their posets of testable properties L_1 and L_2 cannot be isomorphic. This fact is to be interpreted as the impossibility of deriving, in general, the set L of testable properties from the lattice $\mathscr{L}$ of properties.

4.2. The BZ Lattice of Testable Properties

In the previous subsection we have shown that, in a pqs without specific peculiar axioms, the property lattice $\mathscr{L}$ does not carry the whole physical content of the theory. On the other hand, if we assume Piron's axioms C, P, and A or Aerts' axioms 1–5, the resulting theory is not general enough to describe entities which cannot be described by customary quantum mechanics. We then seek for a specific peculiar axiom which allows us to identify the physical content of $\mathscr{L}$ with that of L, but is weaker than the specific peculiar axioms of Piron's and Aerts' theories.

A pqs $\langle \mathscr{S}, \mathscr{Q}; I, \pi, \tilde{}; T\rangle$ is said to be *complete* if the following axiom holds.

Completeness Axiom. Let $\langle \mathscr{S}, \mathscr{Q}; I, \pi, \tilde{}; T\rangle$ be a pqs describing an entity S. For every property $a \in \mathscr{L}$ there exists a (primitive) question $\nu(a) \in a$ such that $\Sigma_T(\nu(a)) = \Sigma_F(\nu(a))^{\#}$ and $\Sigma_F(\nu(a)) = \Sigma_T(\nu(a))^{\#}$.

Example 4.2. The Hilbertian model of a pqs presented in Section 3.4 represents a complete pqs. Indeed, for every Hilbertian question $\alpha \in \mathscr{F}(\mathscr{H})$, the primitive question $\nu([\alpha]_{\approx})$ is the orthogonal projection $E_\alpha(\{1\}) \in \mathscr{E}(\mathscr{H})$.

In this subsection we show that for a complete pqs there exists a canonical procedure ($\mathscr{L} \to L$) which recovers the poset L from the property lattice $\mathscr{L}$.

A trivial consequence of the completeness is that, for any question α, the set of states $\Sigma_T(\alpha)$, which is equal to $\Sigma_F(\alpha)^{\#}$, is a $\#$-closed subset of Σ, i.e., $\mathscr{L}(\Sigma, \mathscr{Q}) \subseteq \mathscr{L}(\Sigma, \#)$, as stated in the following proposition.

Proposition 4.3. Let $\langle \mathscr{S}, \mathscr{Q}; I, \pi, \tilde{}; T\rangle$ be a pqs. If the completeness axiom holds, then for every $\alpha \in \mathscr{Q}$, for every $a \in \mathscr{L}$, and for every $e \in L$:

(i) $\Sigma_T(\alpha), \Sigma_F(\alpha), \Sigma_T(a), \Sigma_T(e), \Sigma_F(e) \in \mathscr{L}(\Sigma, \#)$, i.e.,

$$\mathscr{L}(\Sigma, \mathscr{Q}) \subseteq \mathscr{L}(\Sigma, \#)$$

(ii) $\Sigma_T(\alpha)(\#)\Sigma_F(\alpha)$ and $\Sigma_T(e)(\#)\Sigma_F(\alpha)$, i.e., $L(\Sigma, \mathscr{Q}) \subseteq L(\Sigma, \#)$.

The following Propositions 4.4 and 4.5 and Theorem 4.1 show that in a complete pqs the equalities $\mathscr{L}(\Sigma, \mathscr{Q}) = \mathscr{L}(\Sigma, \#)$ and $L(\Sigma, \mathscr{Q}) = L(\Sigma, \#)$ hold.

Proposition 4.4. Let $\langle \mathscr{S}, \mathscr{Q}; I, \pi, \tilde{}; T\rangle$ be a complete pqs. Then, the following statements hold.

(i) For every question $\alpha \in \mathscr{Q}$, $\Sigma_F(\alpha) \subseteq \Sigma_F(\nu([\alpha]_{\approx}))$.
(ii) For every state $p \in \Sigma$, $\Sigma_F(\nu(p)) = \{p\}^{\#}$.

Proof. (i) For every $q \in \Sigma_F(\alpha)$ we have $q\#p$ for $p \in \Sigma_T(\alpha)$; then

$$\Sigma_F(\alpha) \subseteq \Sigma_T(\alpha)^{\#} = \Sigma_T(\nu([\alpha]))^{\#} = \Sigma_F(\nu([\alpha]_{\approx}))$$

(ii) Let us denote by $\mathscr{A}$ the set of all questions α such that $\Sigma_T(\alpha) = \Sigma_T(\nu(p))$, and by $\mathscr{B}$ the set of the questions β such that $p \in \Sigma_T(\beta)$. Then we have:

(a) $\Sigma_F(\nu(p)) = \bigcup_{\alpha \in \mathscr{A}} \Sigma_F(\alpha)$.
(b) $\{p\}^{\#} = \bigcup_{\beta \in \mathscr{B}} \Sigma_F(\beta)$.

Now, for every $\beta \in \mathscr{B}$, we have $\Sigma_F(\beta) \subseteq \Sigma_F(\nu(p))$. Indeed, $\beta \in \mathscr{B}$ iff $p \in \Sigma_T(\beta)$. On the other hand,

$$\Sigma_T(\nu(p)) = \bigcap_{\beta \in \mathscr{B}} \Sigma_T(\beta)$$

so that for any $\beta \in \mathscr{B}$, $\Sigma_T(\nu(p)) \subseteq \Sigma_T(\beta) = \Sigma_T(\nu(\beta))$; this implies $\Sigma_F(\beta) \subseteq \Sigma_F(\nu([\beta]_\approx)) \subseteq \Sigma_F(\nu(p))$ [since $\Sigma_T(\nu([\beta]_\approx))^\# \subseteq \Sigma_T(\nu(p))^\#$].

Therefore,

$$\{p\}^\# = \bigcup_{\beta \in \mathscr{B}} \Sigma_F(\beta) \subseteq \Sigma_T(\nu([\beta]_\approx))$$

On the other hand, for every $\alpha \in \mathscr{A}$,

$$\Sigma_T(\alpha) = \Sigma_T(\nu(p)) = \bigcap_{p \in \Sigma_T(\beta)} \Sigma_T(\beta)$$

which implies $P \in \Sigma_T(\alpha)$, i.e., $\alpha \in \mathscr{B}$, and so

$$\bigcup_{\alpha \in \mathscr{A}} \Sigma_F(\alpha) \equiv \Sigma_F(\nu(p)) \subseteq \bigcup_{\alpha \in \mathscr{A}} \Sigma_F(\alpha) \quad \bigcup_{\alpha \in \mathscr{B} \setminus \mathscr{A}} \Sigma_F(\alpha) = \{p\}^\#$$

Thus $\Sigma_F(\nu(p)) = \{p\}^\#$. ■

Proposition 4.5. If the completeness axiom holds, then the following statements hold.

(i) For every $p \in \Sigma$,

$$\Sigma_T(\nu(p))^\# = \{p\}^{\#\#}$$

(ii) For every $A \subseteq \Sigma$, there exists a question α such that

$$\Sigma_T(\alpha) = A^{\#\#} \qquad \text{and} \qquad \Sigma_F(\alpha) = A^\#$$

(iii) For every $a \in \mathscr{L}$, $\Sigma_T(a) \in \mathscr{L}(\Sigma, \#)$ and the mapping

$$\mathrm{ext}_1: \quad \mathscr{L} \mapsto \mathscr{L}(\Sigma, \#), \quad a \mapsto \mathrm{ext}_1(a) = \Sigma_T(a)$$

is an isomorphism between the two complete lattices $\mathscr{L}$ and $\mathscr{L}(\Sigma, \#)$.

Proof. (i) This is a trivial consequence of Proposition 4.4(ii).

(ii) We consider the question $\beta = \pi_{p \in \mathscr{A}^\#} \nu(p)^\sim$; we have

$$\Sigma_T(\beta) = \bigcap_{p \in \mathscr{A}^\#} \Sigma_F(\nu(p)) \bigcap_{p \in A^\#} \{p\}^\# = A^{\#\#} \qquad \text{[Proposition 4.4(ii)]}$$

We now define the question $\alpha = \nu(\beta)$; then $\Sigma_T(\alpha) = \Sigma_T(\nu(\beta)) = A^{\#\#}$ and $\Sigma_F(\alpha) = \Sigma_T(\nu([\beta]_\approx)) = A^\#$.

(iii) This follows from $\mathscr{L}(\Sigma, \mathscr{Q}) = \mathscr{L}(\Sigma, \#)$. ■

An immediate, but important, consequence of Proposition 4.5(iii) is that the orthocomplementation $A \mapsto A^{\#}$ (Proposition 3.3) of $\mathscr{L}(\Sigma, \#)$ induces an orthocomplementation $'$ on the property lattice, $\mathscr{L}$, given by

$$': \quad \mathscr{L} \mapsto \mathscr{L}, \qquad a \mapsto a' = \mathrm{ext}_1^{-1}(\mathrm{ext}_1[a]^{\#})$$

Finally, we state the equality $L(\Sigma, \mathscr{Q}) = L(\Sigma, \#)$, which implies the isomorphism $L \leftrightarrow L(\Sigma, \#)$.

Theorem 4.1. Let $\langle \mathscr{S}, \mathscr{Q}; I, \pi, {}^{\sim}; T \rangle$ be a pqs and assume that the completeness axiom holds. Then, for every pair (A, B) of $\#$-closed subsets of Σ, i.e., $A, B \in \mathscr{L}(\Sigma, \#)$, with $A(\#)B$, there exists a question γ such that $\Sigma_T(\gamma) = A$ and $\Sigma_F(\gamma) = B$, i.e., $L(\Sigma, \mathscr{Q}) = L(\Sigma, \#)$. Therefore, $L(\Sigma, \mathscr{Q})$ is a complete BZ-lattice, too.

Proof. For every pair $(A, B) \in \mathscr{L}^2(\Sigma, \#)$, with $A(\#)B$, and in view of Propositions 4.3–4.5, there exist two primitive questions α and β such that

$$\Sigma_T(\alpha) = A \qquad \text{and} \qquad \Sigma_T(\beta) = B$$

Then we form the question

$$\gamma = \alpha \cdot \beta^{\sim}$$

We have

$$\begin{aligned} \Sigma_T(\gamma) &= \Sigma_T(\alpha) \cap \Sigma_F(\beta) \\ &= A \cap B^{\#} \qquad &&\text{(by Definition 3.2)} \\ &= A &&\text{(since } A \subseteq B^{\#}) \end{aligned}$$

On the other hand,

$$\begin{aligned} \Sigma_F(\gamma) &= \Sigma_F(\alpha) \cap \Sigma_T(\beta) \\ &= A^{\#} \cap B \qquad &&\text{(by Definition 3.2)} \\ &= B &&\text{(since } B \subseteq A^{\#}) \qquad \blacksquare \end{aligned}$$

Theorem 4.1 states that, as a consequence of the equality $L(\Sigma, \mathscr{Q}) = L(\Sigma, \#)$, the mapping

$$\mathrm{ext}_2: \quad L \mapsto L(\Sigma, \#), \qquad \mathrm{ext}_2(e) = (\Sigma_T(e), \Sigma_F(e))$$

is an isomorphism between complete BZ-lattices. Now, from the definition of $L(\Sigma, \#)$, we see that it is completely determined by the orthocomplemented complete lattice $\mathscr{L}(\Sigma, \#)$. Since the latter is isomorphic to the property lattice $\mathscr{L}$ and the former is isomorphic to the BZ-lattice L of testable properties, we can assert that L is completely determined by $\mathscr{L}$. Indeed, if $\langle \mathscr{S}, \mathscr{Q}; I, \pi, {}^{\sim}; T \rangle$ is a complete pqs and $\mathscr{L}$ is the corresponding property lattice, we define

$$L(\mathscr{L}) = \{(a, b) \in \mathscr{L}^2 | a \leq b'\}$$

and the partial order relation $(a, b) \preceq (c, d)$ iff $a \leq c$ and $d \leq b$ with respect to which $L(\mathscr{L})$ is a complete BZ-lattice isomorphic to L, the isomorphism being given by the mapping

$$L(\mathscr{L}) \mapsto L, \qquad (a, b) \mapsto e \in L \qquad \text{such that} \quad \mathrm{ext}_2(e) = (\mathrm{ext}_1(a), \mathrm{ext}_1(b))$$

Hence, for every complete pqs there exists a canonical procedure

$$\mathscr{L} \to L(\mathscr{L}) \leftrightarrow L \qquad (\mathscr{L} \to L)$$

to recover the BZ-lattice of testable properties L from the property lattice $\mathscr{L}$. Thus, we may assert that *in a complete pqs, the physical content carried by the property lattice $\mathscr{L}$ coincides with the physical content carried by the BZ-lattice of testable properties L.*

4.3. Separated Entities

In the preceding subsection, we have seen that, for a complete pqs, the physical content carried by the BZ-lattice L is identical to the one carried by the property lattice $\mathscr{L}$ of JP theory. In this subsection we use Aerts' (1981*a*, 1982) theory of separated entities to prove that, if two separated entities are individually described by complete pqs, then the entity consisting of the two entities cannot be described, in general, by a complete pqs.

Note. In the sequel, axiom 1 of Aerts' theory (Section 3.3) will be simply referred to as axiom 1.

Our argument needs the following further notions about pqs.

(a) *Performable-together questions.* Let us quote Aerts (1982):

> We shall say that we can perform both questions α and β together iff there exists an experiment $E(\alpha, \beta)$ having four outcomes that we shall label by {yes, yes}, {yes, no}, {no, yes}, and {no, no}, and such that
>
> α is true iff we are certain to get one of the outcomes {yes, yes} or {yes, no} for the experiment E.
>
> $\alpha^\sim$ is true iff we are certain to get one of the outcomes {no, yes} or {no, no} for E.
>
> β is true iff we are certain to get one of the outcomes {yes, yes} or {no, yes} for E.
>
> $\beta^\sim$ is true iff we are certain to get one of the outcomes {yes, no} or {no, no} for E.

If α and β are performable-together questions, the question $\alpha \triangle \beta$ (resp., $\alpha \triangledown \beta$, $\alpha \ominus \beta$) consists in performing $E(\alpha, \beta)$ and attributing the outcome "y" to $\alpha \triangle \beta$ (resp., $\alpha \triangledown \beta$, $\alpha \ominus \beta$) if the outcome (y, y) (resp., $[(y, y)$ or (y, n) or $(n, y)]$, $[(y, y)$ or $(n, n)]$) is obtained for $E(\alpha, \beta)$; in the other cases we attribute the outcome "n" to $\alpha \triangle \beta$ (resp., $\alpha \triangledown \beta$, $\alpha \ominus \beta$).

(b) *Separated questions and entities*: "We shall say that two questions α and β of an entity S that can be performed together are separated iff, when for an arbitrary state of the entity there is a certain chance to obtain one answer for α and another one for β, then there is for this state of the entity a certain chance to obtain this combination for $E(\alpha, \beta)$ .... If we have an entity S consisting of two entities S_1 and S_2, then S_1 and S_2 are said to be separated iff every question of S_1 is separated from every question of S_2" (Aerts, 1982).

As proved by Aerts (1982), if α and β are two performable-together questions, then:

(i) $\Sigma_T(\alpha \triangle \beta) = \Sigma_T(\alpha) \cap \Sigma_T(\beta)$; from which we have that

$$\alpha \triangle \beta \in a \wedge b$$

where one denotes by a and b the corresponding properties.

(ii) $\Sigma_T(\alpha) \cup \Sigma_T(\beta) \subseteq \Sigma_T(\alpha \triangledown \beta)$ and the property c generated by $\alpha \triangledown \beta$ is an upper bound of the set of property $\{a, b\}$.

Furthermore, if α and β are separated, then:

(iii) $\Sigma_T(\alpha \triangledown \beta) = \Sigma_T(\alpha) \cup \Sigma_T(\beta)$, from which it follows that

$$\alpha \triangledown \beta \in a \vee b$$

A formalization of the notion of performable-together questions [item (a) above] may be found in Cattaneo and Nisticò (1989*b*), in which the following propositions are proved.

Proposition 4.6. Let α and β be two questions performable together, and let us denote the set

$$\{\alpha, \beta, \alpha^{\sim}, \beta^{\sim}, \alpha \triangle \beta, \alpha \triangledown \beta, \alpha \ominus \beta\}$$

by $\mathscr{H}(\alpha, \beta)$. If $\gamma_1, \gamma_2 \in \mathscr{H}(\alpha, \beta)$, then γ_1 and γ_2 are performable together. Furthermore, α true implies $\alpha \triangledown \beta$ true.

Proposition 4.7. If axiom 1 holds and α and β are performable-together primitive questions, then also $\alpha \triangle \beta$, $\alpha \triangledown \beta$, and $\alpha \ominus \beta$ are primitive questions. Furthermore, (i) $\alpha^{\sim} \triangle \beta^{\sim}$ true implies $(\alpha \triangledown \beta)^{\sim}$ true, (ii) $\alpha^{\sim}$ true implies $(\alpha \triangle \beta)^{\sim}$ true.

Let us now consider two separated entities S_1 and S_2, with sets of questions $\mathscr{Q}_1$ and $\mathscr{Q}_2$, respectively. According to Aerts' theory, the set $\mathscr{Q}$ of the questions of entity S constituted by the separated entities S_1 and S_2 consists of questions α of the form

$$\alpha = \alpha_1 \cdot \alpha_2 \cdot \pi_i \alpha_1^i \triangle \alpha_2^i \cdot \pi_j \alpha_1^j \triangledown \alpha_2^j \cdot \pi_k \alpha_1^k \ominus \alpha_2^k \tag{1}$$

where the questions indexed by 1 (resp. 2) pertain to S_1 (resp. S_2).

We can now present our counterexample. Suppose that we have to describe two separated entities S_1 and S_2 such that entity S_k $(k=1, 2)$ has two primitive, nontrivial questions α_k and β_k, which can be performed together. Furthermore, suppose that there exist four different states for S_k, $q_1^{(k)}$, $q_2^{(k)}$, $q_3^{(k)}$, and $q_4^{(k)}$, such that questions $\alpha_k \triangle \beta_k$, $\alpha_k \triangle \beta_k^{\sim}$, $\alpha_k^{\sim} \triangle \beta_k$, and $\alpha_k^{\sim} \triangle \beta_k^{\sim}$ are true, respectively. This is all that is known about S_k, and thus we have to construct the theory for S_k only on the basis of this information.

Remark 3. Such entities are quite usual in physics. For instance, we may think of S_k as an electron with only two energy levels E_1 and E_2 and for which the only questions available are α_k and β_k, with α_k being "the spin of S_k along the z axis is equal to $1/2\hbar$," and β_k being "S_k has energy equal to E_2."

Let us now build up the question sets of S_k. In accordance with Aerts (1982), we can set

$$\mathcal{Q}_k = \{\pi_i \alpha_k^{(i)} | \{\alpha_k^{(i)}\} \subseteq \mathcal{G}_k\}, \qquad k=1, 2 \tag{2}$$

where $\mathcal{G}_k$ is the set of the primitive questions of $\mathcal{Q}_k$, i.e.,

$$\begin{aligned}\mathcal{G}_k(\alpha_k, \beta_k) = \{&O, \alpha_k \triangle \beta_k, \alpha_k \triangle \beta_k^{\sim}, \alpha_k^{\sim} \triangle \beta_k, \alpha_k^{\sim} \triangle \beta_k^{\sim}, \alpha_k, \beta_k,\\ &\alpha_k \ominus \beta_k, (\alpha_k \ominus \beta_k)^{\sim},\\ &\beta_k^{\sim}, \alpha_k^{\sim}, \alpha_k \nabla \beta_k, (\alpha_k^{\sim} \triangle \beta_k)^{\sim}, (\alpha_k \triangle \beta_k^{\sim})^{\sim}, (\alpha_k \triangle \beta_k)^{\sim}, I\}\end{aligned}$$

We stress that, by construction, both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ satisfy the completeness axiom and axiom 1.

Theorem 19 in Aerts (1982) implies that in the state $p_1 = q_2^{(1)} \wedge q_2^{(2)}$, the question $(\alpha_1 \triangle \beta_1^{\sim}) \triangle (\alpha_2 \triangle \beta_2^{\sim})$ is true. Analogously, there exist four states, denoted by p_2, p_3, p_4, and p_5, such that the four questions

$$(\alpha_1 \triangle B_1^{\sim}) \triangle (\alpha_2^{\sim} \triangle \beta_2)$$

$$(\alpha_1^{\sim} \triangle \beta_1) \triangle (\alpha_2 \triangle \beta_2^{\sim})$$

$$(\alpha_1^{\sim} \triangle \beta_1) \triangle (\alpha_2^{\sim} \triangle \beta_2)$$

and

$$(\alpha_1^{\sim} \triangle \beta_1^{\sim}) \triangle (\alpha_2 \triangle \beta_2^{\sim})$$

are true, respectively. It follows that

$$\{p_1, p_2, p_3\} \# \{p_4, p_5\} \tag{3}$$

Our goal is to show that $\mathcal{Q}$ is not complete. We proceed by contradiction. If $\mathcal{Q}$ were complete, there should exist a primitive question $\alpha \in \mathcal{Q}$, i.e., of the form (1), such that the following statements hold.

(I) In each state p_1, p_2, or p_3 the question α is true.
(II) In each state p_4 or p_5 the question $\alpha^{\sim}$ is true.

If α had the form (1), then Theorem 22 of Aerts (1982) would apply, so that α must have one of the forms δ_1, δ_2, $\delta_1 \triangle \delta_2$, $\delta_1 \nabla \delta_2$, or $\delta_1 \ominus \delta_2$, where δ_1 and δ_2 are primitive questions of $\mathscr{Q}_1$ and $\mathscr{Q}_2$, respectively, i.e., $\delta_1 \in \mathscr{G}_1$ and $\delta_2 \in \mathscr{G}_2$. But statements (I) and (II) do not hold for any of these five questions, as we are going to show. In Cattaneo and Nisticò (1989*b*) we proved the following propositions.

Proposition 4.8. Let $\delta_k \in \mathscr{G}_k(\alpha_k, \beta_k)$ and let p be a state. Then either

$$p \in \Sigma_T(\alpha_k) \cap \Sigma_T(\beta_k)$$

implies $p \in \Sigma_T(\delta_k)$ or

$$p \in \Sigma_T(\alpha_k) \cap \Sigma_T(\beta_k)$$

implies $p \in \Sigma_F(\delta_k)$.

Proposition 4.9. If axiom 1 holds, $\delta_k \in \mathscr{G}_k(\alpha_k, \beta_k)$ iff there exists $\delta'_k \in \mathscr{G}_k(\alpha_k^{\sim}, \beta_k)$ such that $\Sigma_T(\delta_k) = \Sigma_T(\delta'_k)$ and, therefore, $\Sigma_F(\delta_k) = \Sigma_F(\delta'_k)$.

By using these results, in Cattaneo and Nisticò (1990) we proved that if $\delta_1 \in \mathscr{G}_1(\alpha_1, \beta_1)$ and $\delta_2 \in \mathscr{G}_2(\alpha_2, \beta_2)$, then $\alpha \in \mathscr{G}(\delta_1, \delta_2)$ and statements (I)–(II) above imply

$$\begin{Bmatrix} p_1 \in \Sigma_T(\delta_1) \\ p_3 \in \Sigma_T(\delta_1) \\ p_5 \in \Sigma_F(\delta_1) \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} p_1 \in \Sigma_F(\delta_1) \\ p_3 \in \Sigma_F(\delta_1) \\ p_5 \in \Sigma_T(\delta_1) \end{Bmatrix} \tag{4}$$

Now, we have also $p_2, p_4 \in \Sigma_T(\alpha_2^{\sim}) \cap \Sigma_T(\beta_2)$, and this implies $p_2, p_4 \in \Sigma_T(\delta_2)$ or $p_2, p_4 \in \Sigma_F(\delta_2)$. Then, from Propositions 4.8 and 4.9, it follows that

$$\text{either} \quad \begin{Bmatrix} p_2 \in \Sigma_T(\delta_1) \\ p_4 \in \Sigma_F(\delta_1) \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} p_2 \in \Sigma_F(\delta_1) \\ p_4 \in \Sigma_T(\delta_1) \end{Bmatrix} \tag{5}$$

From (4) and Proposition 4.8, since

$$p_1 \in \Sigma_T(\alpha_1) \cap \Sigma_T(\beta_1^{\sim})$$

and

$$p_3 \in \Sigma_T(\alpha_1^{\sim}) \cap \Sigma_T(\beta_1)$$

it follows that

$$\begin{aligned} &\text{either} \quad \begin{Bmatrix} q \in \Sigma_T(\alpha_1) \cap \Sigma_T(\beta_1^{\sim}) \text{ implies } q \in \Sigma_T(\delta_1) \\ q \in \Sigma_T(\alpha_1^{\sim}) \cap \Sigma_T(\beta_1) \text{ implies } q \in \Sigma_T(\delta_1) \end{Bmatrix} \\ &\text{or} \quad \begin{Bmatrix} q \in \Sigma_T(\alpha_1) \cap \Sigma_T(\beta_1^{\sim}) \text{ implies } q \in \Sigma_F(\delta_1) \\ q \in \Sigma_T(\alpha_1^{\sim}) \cap \Sigma_T(\beta_1) \text{ implies } q \in \Sigma_F(\delta_1) \end{Bmatrix} \end{aligned} \tag{6}$$

On the other hand, from the fact that $p_2 \in \Sigma_T(\alpha_1) \cap \Sigma_T(\beta_1^\sim)$ and $p_4 \in \Sigma_T(\alpha_1^\sim) \cap \Sigma_T(\beta_1)$, and (5) we get

$$\begin{aligned} &\text{either} \quad \begin{Bmatrix} q \in \Sigma_T(\alpha_1) \cap \Sigma_T(\beta_1^\sim) \text{ implies } q \in \Sigma_T(\delta_1) \\ q \in \Sigma_T(\alpha_1^\sim) \cap \Sigma_T(\beta_1) \text{ implies } q \in \Sigma_F(\delta_1) \end{Bmatrix} \\ &\text{or} \quad \begin{Bmatrix} q \in \Sigma_T(\alpha_1) \cap \Sigma_T(\beta_1^\sim) \text{ implies } q \in \Sigma_F(\delta_1) \\ q \in \Sigma_T(\alpha_1^\sim) \cap \Sigma_T(\beta_1) \text{ implies } q \in \Sigma_T(\delta_1) \end{Bmatrix} \end{aligned} \tag{7}$$

We see that (6) and (7) form a contradiction. Even though the completeness axiom holds for both $\mathcal{Q}_1$ and $\mathcal{Q}_2$, the same axiom cannot hold for $\mathcal{Q}$. Indeed, we have that p_1, p_2, p_3 are preclusive to p_4, p_5, but no question α exists in $\mathcal{Q}$ such that p_1, p_2, $p_3 \in \Sigma_T(\alpha)$ and $p_4, p_5 \in \Sigma_F(\alpha)$; then the pqs $\langle \mathcal{S}, \mathcal{Q}; I, \pi, {}^\sim; T \rangle$ which, according to Aerts, describes two separated entities as a unique entity, in general is not complete.

Remark 4. In Cattaneo and Nisticò (1990) doubts are raised about the empirical adequateness of Aerts' theory of separated entities. This does not entail our argument, because Aerts' theory is perfectly self-consistent.

REFERENCES

Aerts, D. (1981*a*). The one and the many. Towards a unification of the quantum and the classical description of one and many physical entities, Doctoral thesis, Vrije Universiteit Brussel, TENA.

Aerts, D. (1981*b*). Description of compound physical systems and logical interaction of physical systems, in *Current Issues in Quantum Logic*, E. G. Beltrametti and B. C. van Fraassen, eds., Plenum Press, New York.

Aerts, D. (1982). Description of many and separated physical entities without the paradoxes encountered in quantum mechanics, *Foundations of Physics*, **12**, 1131.

Aerts, D., and Daubechies, I. (1978*a*). About the structure-preserving maps of a quantum mechanical propositional system, *Helvetica Physica Acta*, **51**, 637.

Aerts, D., and Daubechies, I. (1978*b*). Physical justification for using the tensor product to describe two quantum systems as one joint system, *Helvetica Physica Acta*, **51**, 661.

Aerts, D., and Daubechies, I. (1979). A characterization of subsystems in physics, *Letters on Mathematical Physics*, **3**, 11.

Cattaneo, G., and Nisticò, G. (1989*a*). Brouwer–Zadeh posets and three-valued Łukasiewicz posets, *Fuzzy Sets and Systems*, **33**, 165.

Cattaneo, G., and Nisticò, G. (1989*b*). Coexistence of questions in Jauch–Piron theory, Preprint DMUC 1989, Dipartimento di Scienze dell'Informazione, Università di Milano, Milan, Italy.

Cattaneo, G., and Nisticò, G. (1990). A note on Aerts' description of separated entities, *Foundations of Physics*, **20**, 119.

Cattaneo, C., dalla Pozza, C., Garola, C., and Nisticò, G. (1988). On the logical foundation of the Jauch–Piron approach to quantum physics, *International Journal of Theoretical Physics*, **27**, 1313.

Jauch, J. A. (1983). *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

Jauch, J. M., and Piron, C. (1969). On the structure of quantal propositions systems, *Helvetica Physica Acta*, **42**, 842.

Nash, C. G., and Joshi, G. C. (1987*a*). Composite systems in quaternionic quantum mechanics, *Journal of Mathematical Physics*, **28**, 2883.

Nash, C. G., and Joshi, G. C. (1987*b*). Component states of a composite quaternionic system, *Journal of Mathematical Physics*, **28**, 2886.

Piron, C. (1976). *Foundations of Quantum Physics*, Benjamin, Reading, Massachusetts.

Pulmannová, S. (1985). Tensor product of quantum logics, *Journal of Mathematical Physics*, **26**, 1.

Raggio, G. A. (1981). States and composite systems in W^*-algebraic quantum mechanics, ETH Zurich, preprint No. 6824.

Von Neumann, J. (1955). *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, New Jersey.